Exam I
Section I
Part A — No Calculators

1. A p. 1

$y = f(x)$ is decreasing if and only if $f'(x) < 0$
$f'(x) = \ln(x-2) < 0$ occurs if and only if $0 < x - 2 < 1$.
This happens if and only if $2 < x < 3$.

2. B p. 1

$$y = x\cos x$$
$$\frac{dy}{dx} = \cos x - x\sin x$$
$$\cos x - x\sin x = 0$$
$$\cos x = x\sin x$$
$$\frac{1}{x} = \tan x$$

3. E p. 2

With $F(x) = G[x + G(x)]$, the Chain Rule gives
$F'(x) = G'[x + G(x)] \cdot (1 + G'(x))$

Then $F'(1) = G'[1 + G(1)] \cdot (1 + G'(1))$

From the graph of the function G, we find $G(1) = 3$.

Hence $F'(1) = G'[1+3] \cdot (1 + G'(1))$
$= G'(4) \cdot (1 + G'(1))$

From the graph of G we can determine that $G'(4) = \frac{2}{3}$ and $G'(1) = -2$.

Thus, $F'(1) = \frac{2}{3} \cdot (-1) = -\frac{2}{3}$.

4. C p. 2

$$\int_2^6 \left[\frac{1}{x} + 2x \right] dx = \ln|x| + x^2 \Big]_2^6$$
$$= (\ln 6 + 36) - (\ln 2 + 4) = \ln\frac{6}{2} + 32 = \ln 3 + 32$$

5. E p. 2

$$f(x) = \frac{(\ln x)^2}{x}$$

$$f'(x) = \frac{x \cdot 2(\ln x) \cdot \frac{1}{x} - (\ln x)^2}{x^2} = \frac{(\ln x) \cdot (2 - \ln x)}{x^2}$$

The critical numbers are $x = 1$ and $x = e^2$.

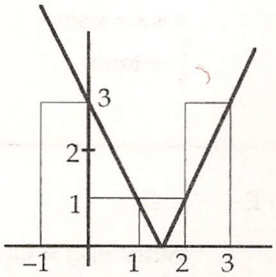
$x > e^2 \Rightarrow f'(x) < 0 \Rightarrow$ f is decreasing.
$1 < x < e^2 \Rightarrow f'(x) > 0 \Rightarrow$ f is increasing.
$0 < x < 1 \Rightarrow f'(x) < 0 \Rightarrow$ f is decreasing.

The relative maximum is at $x = e^2$.

---

6. D p. 3

Graph the function $f(x) = |2x - 3|$ on the interval $[-1,3]$. Since the interval has length 4 and the Riemann sum is to have 4 equal subdivisions, each subdivision has length 1. Since it is to be a right-hand Riemann sum, we use function values at the right-hand ends of the intervals; that is, at $x = 0, 1, 2,$ and 3.



$$R_4 = 1 \cdot [f(0) + f(1) + f(2) + f(3)]$$
$$= 1 \cdot [3 + 1 + 1 + 3] = 8$$

---

7. A p. 3

$y = x^3 + 3x^2 + 2 \Rightarrow \frac{dy}{dx} = 3x^2 + 6x$

$\frac{d^2y}{dx^2} = 6x + 6 = 0 \Rightarrow x = -1 \Rightarrow \begin{cases} y = 4 \\ \frac{dy}{dx} = -3 \end{cases}$

Hence the point of inflection is $(-1,4)$ and the slope of the tangent is -3. Then the equation of the tangent is $y - 4 = -3(x + 1)$, so $y = -3x + 1$.

---

8. D p. 3

$$\int \cos(3 - 2x)\,dx = -\tfrac{1}{2} \int (-2)\cos(3 - 2x)\,dx$$
$$= -\tfrac{1}{2} \sin(3 - 2x) + C$$

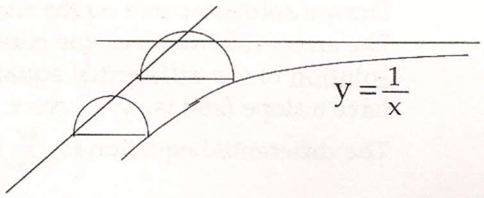
9. B p. 4

$$\lim_{x\to\infty} \frac{\sqrt{9x^2+2}}{4x+3} = \lim_{x\to\infty} \frac{x\sqrt{9+\frac{2}{x^2}}}{4x+3} = \lim_{x\to\infty} \frac{\sqrt{9+\frac{2}{x^2}}}{4+\frac{3}{x}} = \frac{3}{4}$$

10. B p. 4

Each cross section perpendicular to the x-axis (at coordinate x) is a semicircle of radius $\frac{1}{2x}$. The cross-sectional area is $\frac{1}{2}\pi\left(\frac{1}{2x}\right)^2$.

Hence the volume of the solid is given by:



$$V = \int_1^4 \frac{1}{2}\pi\left(\frac{1}{2x}\right)^2 dx = \frac{\pi}{8}\int_1^4 \frac{1}{x^2}\,dx = \frac{\pi}{8}\left(-\frac{1}{x}\right]_1^4 = \frac{\pi}{8}\left(-\frac{1}{4}+1\right) = \frac{3\pi}{32}.$$

11. A p. 4

$f(x) = \ln x + e^{-x} \quad \Rightarrow \quad f'(x) = \frac{1}{x} - e^{-x}$

Since $f'(1)$ exists, (C) is False.

Since $f'(1) \neq 0$, (D) and (E) are False.

Since $f'(1) = 1 - \frac{1}{e} > 0$, (A) is True and (B) is False.

12. E p. 5

We are given $F(x) = \displaystyle\int_0^{x^2} \frac{1}{2+t^3}\,dt$.

If we define $G(x) = \displaystyle\int_0^{x} \frac{1}{2+t^3}\,dt$, then $F(x) = G(x^2)$.

By the Chain Rule, $F'(x) = G'(x^2) \cdot (2x)$.

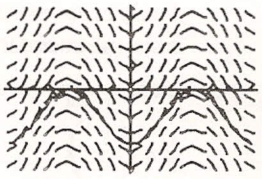
By the Fundamental Theorem, $G'(x) = \frac{1}{2+x^3}$, so that $G'(x^2) = \frac{1}{2+(x^2)^3}$.

Then $F'(x) = \frac{1}{2+(x^2)^3} \cdot (2x)$, and finally $F'(-1) = \frac{1}{2+1} \cdot (-2) = -\frac{2}{3}$.

13. C p. 5

$$\frac{1}{1-(-1)}\int_{-1}^{1}(2t^3-3t^2+4)\,dt = \frac{1}{2}\left[\frac{t^4}{4}-t^3+4t\right]_{-1}^{1}$$
$$= \frac{1}{2}\left[\left(\frac{1}{4}-1+4\right)-\left(\frac{1}{4}+1-4\right)\right] = \frac{1}{2}\cdot 6 = 3$$

14. B p. 5



Draw a solution curve on the slope field. This looks like an up-side down cosine curve. That is, the solution of the differential equation for which we have a slope field is $y = -\cos x$.
The differential equation is $\frac{dy}{dx} = \sin x$.

15. B p. 6

$$\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1}\frac{(\sqrt{x}-1)(\sqrt{x}+1)}{(x-1)(\sqrt{x}+1)}$$
$$= \lim_{x\to 1}\frac{x-1}{(x-1)(\sqrt{x}+1)} = \lim_{x\to 1}\frac{1}{\sqrt{x}+1} = \frac{1}{2}$$

16. E p. 6

$$y = \cos^2 x - \sin^2 x$$
$$y' = -2\cos x\sin x - 2\sin x\cos x = -4\sin x\cos x$$

17. C p. 6

$$\int_1^2\left(4x^3+6x-\frac{1}{x}\right)dx = \left(x^4+3x^2-\ln|x|\right]_1^2$$
$$= (16+12-\ln 2)-(1+3-\ln 1) = (24-\ln 2)$$

18. C p. 7

$$\int\frac{x-2}{x-1}\,dx = \int\frac{(x-1)-1}{x-1}\,dx = \int\left[1-\frac{1}{x-1}\right]dx = x-\ln|x-1|+C$$

19. B p. 7

The property that $g(-x) = g(x)$ for all x means that the function g is even. Its symmetry around the y-axis guarantees that $g'(-a) = -g'(a)$.
More formally, differentiating the first property gives

$$g'(-x) \cdot (-1) = g'(x).$$

Thus $$g'(-x) = -g'(x).$$

---

20. A p. 7

$y = \text{Arctan}\,\frac{x}{3}$

$y' = \frac{1}{3}\,\frac{1}{1+\frac{x^2}{9}} = \frac{3}{9+x^2}$. This implies that $y'(0) = \frac{1}{3}$.

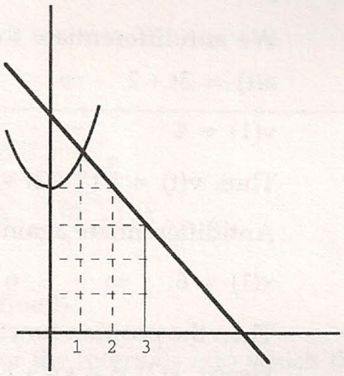
Hence the line goes through the origin with slope $\frac{1}{3}$.

Its equation is $y - 0 = \frac{1}{3}(x-0)$, which can be written $x - 3y = 0$.

---

21. C p. 8

Solution I. With a reasonably careful graph, it is possible to obtain an estimate of the definite integral by counting the squares under the graph of $f(x)$ on the interval $[0,3]$.



Solution II. Having determined that the change in the function definition occurs at $x = 1$, evaluate $\int_0^3 f(x)\,dx$.

This is done in two parts, as:

$$\int_0^3 f(x)\,dx = \int_0^1 (x^2+4)\,dx + \int_1^3 (6-x)\,dx$$

$$= \left[\frac{x^3}{3} + 4x\right]_0^1 + \left[6x - \frac{x^2}{2}\right]_1^3$$

$$= \left[\frac{1}{3} + 4\right] - 0 + \left[18 - \frac{9}{2}\right] - \left(6 - \frac{1}{2}\right) = 12 + \frac{1}{3}$$

---

22. B p. 8

$$\frac{d}{dx}\left(\ln e^{3x}\right) = \frac{d}{dx}\left(3x \ln e\right) = \frac{d}{dx}(3x) = 3$$

23. D p. 8

$g'(x) = 2g(x) \quad \Rightarrow \quad \frac{g'(x)}{g(x)} = 2$

Integrating gives $\ln|g(x)| = 2x + C$

Then $g(x) = e^{2x+C}$

Using the initial condition that $g(-1) = 1$, we have

$$g(-1) = e^{-2+C} = 1 \quad \Rightarrow \quad C = 2$$

Hence $g(x) = e^{2x+2}$

The notation is simpler if we let $y = g(x)$. Then the equation is $y' = 2y$. The solution proceeds as before.

$y' = 2y \quad \Rightarrow \quad \frac{y'}{y} = 2 \quad \Rightarrow \quad \ln|y| = 2x + C$

$\Rightarrow \quad y = \pm e^{2x+C}$ Since $g(-1) = 1$, $y = e^{2x+C}$

---

24. D p. 9

We antidifferentiate the acceleration function to obtain the velocity.

$a(t) = 3t + 2 \quad \Rightarrow \quad v(t) = \frac{3}{2}t^2 + 2t + C$

$v(1) = 4 \quad \Rightarrow \quad 4 = \frac{3}{2} + 2 + C \quad \Rightarrow \quad C = \frac{1}{2}$

Thus $v(t) = \frac{3}{2}t^2 + 2t + \frac{1}{2}$

Antidifferentiate again to obtain $x(t) = \frac{1}{2}t^3 + t^2 + \frac{1}{2}t + D$

$x(1) = 6 \quad \Rightarrow \quad 6 = \frac{1}{2} + 1 + \frac{1}{2} + D \quad \Rightarrow \quad D = 4$

Then the position function is $x(t) = \frac{1}{2}t^3 + t^2 + \frac{1}{2}t + 4$.

Hence $x(2) = 4 + 4 + 1 + 4 = 13$.

---

25. B p. 9

$y = \sqrt{3 + e^x}$ passes through $(0,2)$.

$\frac{dy}{dx} = \frac{e^x}{2\sqrt{3 + e^x}}$; when $x = 0$, this has a value of $\frac{1}{4}$.

The equation of the tangent line at $(0,2)$ is $y - 2 = \frac{1}{4}x$, or $y = 2 + \frac{1}{4}x$.

When $x = 0.08$, $y = 2 + \frac{1}{4}(.08) = 2.02$.

26. B p. 9

For $1 < t < 3$, the leaf rises 5 feet in 2 seconds. $s = \frac{5}{2} = 2.5$ ft/sec.

For $3 < t < 5$, the leaf falls 10 feet in 2 seconds. $s = \frac{10}{2} = 5$ ft/sec.

For $5 < t < 7$, the leaf rises 3 feet in 2 seconds. $s = \frac{3}{2} = 1.5$ ft/sec.

For $7 < t < 9$, the leaf falls 8 feet in 2 seconds. $s = \frac{8}{2} = 4$ ft/sec.

Since the slope of the graph is constant on each of these intervals, the only other interval of interest is $0 < t < 1$. During that period, the leaf rises 1.5 feet in 1 second.

Then $s = \frac{1.5}{1} = 1$ ft/sec.

The maximum speed is 5 ft/sec, occurring in the interval $3 < t < 5$.

---

27. C p. 10

Differentiating the given volume function with respect to t gives

$$\frac{dV}{dt} = \pi(12h - h^2)\frac{dh}{dt}.$$

We know $\frac{dV}{dt} = 30\pi\ \text{ft}^3/\text{sec}$, and are interested in $\frac{dh}{dt}$ when $h = 2$ ft. Substituting these values, we have

$30\pi = \pi(12 \cdot 2 - 2^2)\frac{dh}{dt}$. Hence $\frac{dh}{dt} = \frac{30\pi}{20\pi} = 1.5$ ft/hr.

---

28. E p. 10

$f(x) = 2x^{5/3} - 5x^{2/3} \quad \Rightarrow \quad f'(x) = \frac{10}{3}x^{2/3} - \frac{10}{3}x^{-1/3} = \frac{10}{3}x^{-1/3}(x-1)$

The function f has two critical numbers:

$x = 1$ (where $f'(x) = 0$) and $x = 0$ (where $f'(x)$ is undefined).

To determine the sign of the first derivative, we consider the intervals into which these critical numbers divide the domain of the function.

	$x<0$	$0<x<1$	$x>1$
$x^{-1/3}$	−	+	+
$x-1$	−	−	+
$f'(x)$	+	−	+

$f(x)$ is increasing if and only if $f'(x) > 0$. This occurs if $x < 0$ or $x > 1$.

Exam I
Section I
Part B — Calculators Permitted

1. D p. 11

I. $\lim_{x \to 1} f(x) = -2$ False

II. $\lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = f'(2) = 2$ True

III. $\lim_{x \to -1^+} f(x) = 1 = f(-3)$ True

2. C p. 11

$f(x) = \sin^2 x \quad \Rightarrow \quad f'(x) = 2 \sin x \cos x = \sin(2x)$

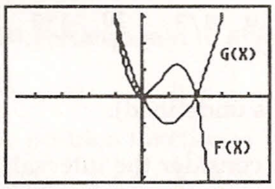
$g(x) = .5\,x^2 \quad \Rightarrow \quad g'(x) = x$

From a calculator graph of the functions f' and g', we see the only possible solution is $x = 0.9$.

3. B p. 12

Here are two possible calculator solutions.

I. First, look at graphs of the functions f and g, and find those intervals where the graph of f is <u>above</u> the graph of g.



The cubic function $f(x)$ is above the quadratic function $g(x)$ when x is between 0 and 2. Thus, the integral of f will have a larger value than the integral of g on the intervals $[0,2]$.

II. Second, use the calculator to evaluate these definite integrals on the intervals $[a,b]$ indicated.

			$\int_a^b f(x)\,dx$	$\int_a^b g(x)\,dx$	
I.	$a = -1$	$b = 0$	.917	1.333	False
II.	$a = 0$	$b = 2$	1.333	-1.333	True
III.	$a = 2$	$b = 3$	-3.583	1.333	False

4. E p. 12

$$y^2 - 3x = 7 \quad \Rightarrow \quad 2y\frac{dy}{dx} - 3 = 0$$

$$\frac{dy}{dx} = \frac{3}{2y}$$

$$\frac{d^2y}{dx^2} = \frac{2y \cdot 0 - 3 \cdot 2\frac{dy}{dx}}{4y^2} = \frac{-6 \cdot \frac{3}{2y}}{4y^2} = -\frac{9}{4y^3}$$

5. D p. 12

I. $h(0) = g(f(0)) = g(5) = 0$ False

II. $h'(x) = g'(f(x)) \cdot f'(x)$

Thus $h'(2) = g'(f(2)) \cdot f'(2)$

$= g'(1) \cdot (-\frac{1}{4}) = (-2) \cdot (-\frac{1}{4}) > 0$ True

III. $h'(4) = g'(f(4)) \cdot f'(4) = g'(2) \cdot 1 = 0 \cdot 1 = 0$ True

6. D p. 13

If (x, e^x) is on the curve, then its distance from the origin is

$D = \sqrt{x^2 + e^{2x}}$. Use a calculator graph of this distance function and find its minimum.

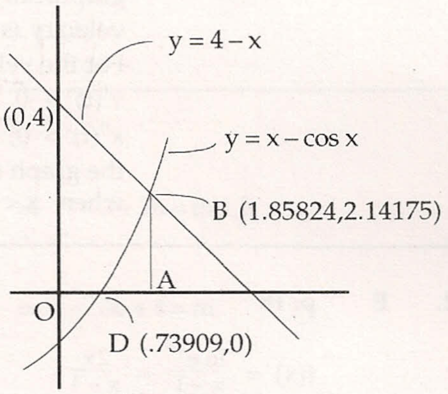
At $x = -0.426$, the minimum distance of 0.78 is achieved.

7. B p. 13

In the figure, we need the area of the region ODBC. This can be calculated as area of trapezoid OABC − area DAB. The coordinates of points B and D are found using the calculator. Then the desired area is

$$a = \int_0^{1.858} (4 - x)\,dx - \int_{.739}^{1.858} (x - \cos x)\,dx$$

$$\approx 4.54.$$



8. C p. 13

$y = 2x + \cos(x^2)$

$y' = 2 - 2x\sin(x^2)$

$y'' = -2\sin(x^2) - 4x^2\cos(x^2)$

Graph the second derivative on the interval $[0,5]$. There are eight zeros at which the sign changes. Each corresponds to an inflection point on the graph of $y = f(x)$.

9. C p. 14

$$\frac{dV}{dt} = \sqrt{1+2^t} \quad \Rightarrow \quad V = \int_0^5 \sqrt{1+2^t}\,dt \approx 14.53 \text{ ft}^3.$$

10. C p. 14

We use the disk (washer) method. $V = \pi \int_1^6 [f(x)]^2\,dx$

Using the Trapezoid Rule with five subintervals to approximate this, we obtain

$$V \approx T_5 = \frac{\pi}{2}[f^2(1) + 2\cdot f^2(2) + 2\cdot f^2(3) + 2\cdot f^2(4) + 2\cdot f^2(5) + f^2(6)]$$
$$= \frac{\pi}{2}[2^2 + 2\cdot 3^2 + 2\cdot 4^2 + 2\cdot 3^2 + 2\cdot 2^2 + 1^2]$$
$$= \frac{\pi}{2}\cdot 81 \approx 127$$

11. E p. 14

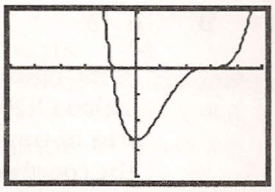
Solution I. We can do the problem algebraically:
Given the position function $x(t) = (t+1)(t-3)^3$,
we differentiate to obtain the velocity function:

$$v(t) = (t+1)\cdot 3(t-3)^2 + (t-3)^3 = 4t\,(t-3)^2$$

For the velocity to be increasing, we need $v'(t) > 0$.

$$v'(t) = (t-3)^2\cdot 4 + (4t)\cdot 2(t-3) = 12(t-3)(t-1).$$

We find that $v'(t) > 0$ if $t > 3$ or $t < 1$.



Solution II. Alternatively, we can do the problem graphically. Given the position $x(t)$, the velocity is $v(t) = x'(t)$.
For the velocity to be increasing, we need $v'(t) > 0$. That is to say, we need $x''(t) > 0$; hence we want the graph of $x(t)$ to be concave up. From the graph of $x(t)$ shown, we recognize that the curve is concave up when $x < 1$ and again when $x > 3$.

12. E p. 15

$$f(x) = \frac{\ln e^{2x}}{x-1} = \frac{2x}{x-1}.$$

The inverse of this function is found by solving $x = \frac{2y}{y-1}$ for y.

$$x = \frac{2y}{y-1} \Rightarrow xy - x = 2y \Rightarrow xy - 2y = x \Rightarrow y(x-2) = x$$
$$\Rightarrow g(x) = y = \frac{x}{x-2}$$

Then $g'(x) = \frac{(x-2)-x}{(x-2)^2} = \frac{-2}{(x-2)^2}$. Hence $g'(3) = -2$.

13. C p. 15

Divide the integrand fraction and rewrite the second term.

$$\int \frac{e^{x^2} - 2x}{e^{x^2}}\, dx = \int \left[1 - \frac{2x}{e^{x^2}} \right] dx \quad = \int \left[1 - 2x\, e^{-x^2} \right] dx$$

$$= \int \left[1 + e^{-x^2}(-2x) \right] dx$$

In the second term of the integrand, the factor $(-2x)$ is the derivative of the exponent in the factor e^{-x^2}. Hence we can perform the antidifferentiation:

$$\int \left[1 + e^{-x^2}(-2x) \right] dx = x + e^{-x^2} + C.$$

14. D p. 16

$$f(x) = (x+2)^5 (x^2-1)^4$$

$$f'(x) = 5(x+2)^4 (x^2-1)^4 + 4(x^2-1)^3 (2x)(x+2)^5$$

$$= (x+2)^4 (x^2-1)^3 \left[5(x^2-1) + 8x(x+2) \right]$$

$$= (x+2)^4 (x+1)^3 (x-1)^3 \left[13x^2 + 16x - 5 \right]$$

The five critical points occur at $x = -2$, $x = \pm 1$, and at the two real solutions of the last quadratic factor. For the latter, $D = b^2 - 4ac = 16^2 - 4(13)(-5) = 516$. Since $D > 0$, there are two real solutions.

15. B p. 16

For continuity, $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) \quad \Rightarrow \quad 3 + 3b = m + b$

For differentiability, $\lim_{x \to 1^-} f'(x) = \lim_{x \to 1^+} f'(x) \quad \Rightarrow \quad 3b + 4 = m$

We solve these two equations simultaneously:

$$\begin{cases} 2b + 3 = m \\ 3b + 4 = m \end{cases} \quad \Rightarrow \quad b = -1 \text{ and } m = 1.$$

16. D p. 17

Solution I. On each two-second time interval, we can approximate the speed by using the average of the speeds at the beginning and the end of the interval.

On the interval $[0,2]$, speed ≈ 33 ft/sec. Distance traveled $\approx$ 66 ft.
On the interval $[2,4]$, speed ≈ 38 ft/sec. Distance traveled $\approx$ 76 ft.
On the interval $[4,6]$, speed ≈ 44 ft/sec. Distance traveled $\approx$ 88 ft.
On the interval $[6,8]$, speed ≈ 51 ft/sec. Distance traveled $\approx$ 102 ft.
On the interval $[8,10]$, speed ≈ 57 ft/sec. Distance traveled $\approx$ 114 ft.
If we add these approximate distances traveled, we obtain 446 ft.

Solution II. Since $v(t) > 0$, on the interval $[0, 10]$, the distance is the value of the integral $\int_0^{10} v(t)\,dt$.

Using Left and Right Riemann Sums, we approximate the integral as follows:

$$L_5 = 2[30 + 36 + 40 + 48 + 54] = 416$$

$$R_5 = 2[36 + 40 + 48 + 54 + 60] = 476$$

$$\text{Distance} = \int_0^{10} v(t)\,dt = \frac{L_5 + R_5}{2} = \frac{416 + 476}{2} = 446$$

17. E p. 17

Rewrite the given formula: $F(x) = -5 + \int_2^x \sin(\frac{\pi t}{4})\,dt$.

We obtain $F'(x)$ by using the Fundamental Theorem:

$$F'(x) = 0 + \sin(\tfrac{\pi x}{4}).$$

We can then evaluate both $F(2)$ and $F'(2)$.

$$F(2) = -5 + \int_2^2 \sin(\tfrac{\pi t}{4})\,dt = -5 + 0 = -5$$

$$F'(2) = \sin(\tfrac{2\pi}{4}) = \sin\tfrac{\pi}{2} = 1.$$

Then $F(2) + F'(2) = -5 + 1 = -4$.

Exam I
Section II
Part A — Calculators Permitted

1. p. 19

(a) Since the graph of f is a straight line here, $f'(3)$ = slope
$= \dfrac{-3-3}{4-2} = -3$.

$2: \begin{cases} 1: \text{difference quotient} \\ 1: \text{answer} \end{cases}$

(b) Using a linear approximation between $(-1,-1)$ and $(1,2)$,
$g'(0) = \dfrac{2-(-1)}{1-(-1)} = \dfrac{3}{2}$.

$2: \begin{cases} 1: \text{difference quotient} \\ 1: \text{answer} \end{cases}$

(c) $h(x) = g[f(x)]$

i) $h(2) = g[f(2)] = g(3) = 0$.

ii) $h'(x) = g'[f(x)] \cdot f'(x)$

$h'(3) = g'[f(3)] \cdot f'(3) = g'(0) \cdot (-3) = \dfrac{3}{2}(-3) = -\dfrac{9}{2}$.

$3: \begin{cases} 1: h(2) \\ 2: h'(3) \end{cases}$

(d) Using areas, we approximate $\int_0^4 f(x)\,dx$ as a trapezoid plus a triangle minus a triangle.

$$\int_0^4 f(x)\,dx = \frac{1}{2}(2)(1+3) + \frac{1}{2}(1)(3) - \frac{1}{2}(1)(3) = 4.$$

2: answer

2. p. 20

(a) To find the coordinates of Q, we write the equation of the tangent to the graph of $y = f(x)$ at the point $P(-2,8)$.

$f'(x) = 3x^2 + 6x - 1$.

Using $x = -2$, we find $f'(-2) = 12 - 12 - 1 = -1$. The line through the point $P(-2,8)$ with slope $m = -1$ is $y - 8 = -1(x + 2)$ which can be rewritten: $y = -x + 6$.

$$3: \begin{cases} 1: \text{slope} \\ 1: \text{tangent equation} \\ 1: \text{intersection pt} \end{cases}$$

We now solve simultaneously the equation of the cubic and the equation of the tangent line.

$$\begin{cases} y = x^3 + 3x^2 - x + 2 \\ y = -x + 6 \end{cases} \Rightarrow x^3 + 3x^2 - x + 2 = -x + 6$$

$$\Rightarrow x^3 + 3x^2 - 4 = 0$$

$$\Rightarrow (x + 2)(x^2 + x - 2) = 0$$

$$\Rightarrow (x + 2)(x + 2)(x - 1) = 0$$

There is the known intersection point where $x = -2$. The new point has an x-coordinate of $x = 1$. The corresponding y-coordinate is $y = 5$. Hence Q is the point (1,5).

(b) To find the inflection point R, we need $f''(x)$.

$f''(x) = 6x + 6$.

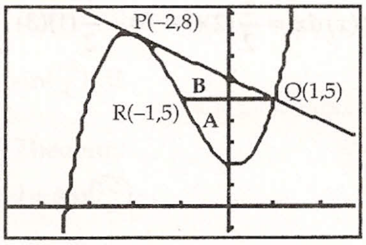
$f''(x) = 0$ if and only if $x = -1$.

When $x = -1$, the cubic function has a y-value of 5.

At $x = -1$, the value of $f''(x)$ changes from negative to zero to positive, hence the point of inflection of the graph of f occurs at $R(-1, 5)$.

$$2: \begin{cases} 1: f''(x) = 0 \\ 1: \text{inflection pt} \end{cases}$$

(c) Shown to the right is a graph of the function f, with the points $P(-2,8)$, $Q(1,5)$, and $R(-1,5)$ identified. To find the areas of the two regions described, we determine the area of the combined region $A \cup B$ and then determine the area of the larger of the two regions, region A.



$$4: \begin{cases} 1: \text{area of region } A \\ 1: \text{area of region } B \\ 2: \text{ratio} \end{cases}$$

Area of region $A \cup B = \int_{-2}^{1} [(-x + 6) - (x^3 + 3x^2 - x + 2\ dx)] = 6.75$.

Area of region $A = \int_{-1}^{1} [5 - (x^3 + 3x^2 - x + 2)]\ dx = 4$.

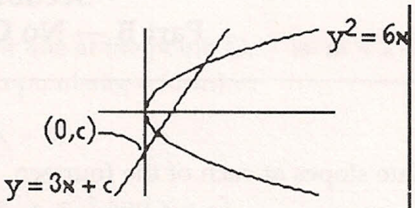
By subtraction, Area of region $B = 2.75$.

The ratio of these areas is $\dfrac{\text{Area of region A}}{\text{Area of region B}} = \dfrac{4}{2.75} = \dfrac{16}{11} = 1.455$.

3. p. 21

(a) Solution I.

Since the line $y = 3x + c$ has slope $m = 3$, we find the point on the curve $y^2 = 6x$ where the tangent has slope 3.



2: answer

Solution II.

Differentiating implicitly, we have $2y \cdot \frac{dy}{dx} = 6$.

Since $\frac{dy}{dx} = 3$, then $y = 1$ and therefore $x = \frac{1}{6}$.

The particular line that passes through $\left(\frac{1}{6}, 1\right)$ is obtained by using those coordinates in $y = 3x + c$. We find $c = \frac{1}{2}$.

We see from the graph above that if c is made smaller than the value that gives tangency, there will be two intersections.

Hence we want $c < \frac{1}{2}$.

Solving the two equations $y^2 = 6x$ and $y = 3x + c$ simultaneously leads to

$$y^2 - 2y + 2c = 0 \qquad (*)$$

This has two solutions if its discriminant is positive.

$4 - 8c > 0 \quad \Rightarrow \quad 4 > 8c \quad \Rightarrow \quad \frac{1}{2} > c.$

(b) Substituting $c = -\frac{3}{2}$ into equation $(*)$ above gives $y^2 - 2y - 3 = 0$.

Thus $(y-3)(y+1) = 0$, so $y = -1$ or 3.

Then $x = \dfrac{y + 3/2}{3}$ and $x = \dfrac{y^2}{6}$ express the curves with x in terms of y.

The area of the region can be written: $\text{area} = \displaystyle\int_{-1}^{3} \left[\frac{y + 3/2}{3} - \frac{y^2}{6}\right] dy$

With a calculator, this is evaluated as $\frac{16}{9} = 1.778$.

$3: \begin{cases} 1: \text{limits} \\ 1: \text{integrand} \\ 1: \text{answer} \end{cases}$

(c) Substituting $c = 0$ into equation $(*)$ gives $y^2 - 2y = 0$.

Thus $y(y-2) = 0$. Hence $y = 0$ or $y = 2$, so $x = 0$ or $x = \frac{2}{3}$.

By the washer method,

$$\text{Vol} = \pi\int_0^{2/3} \left(6x - 9x^2\right) dx = \pi\left[3x^2 - 3x^3\right]_0^{2/3} = \pi\left[3 \cdot \tfrac{4}{9} - 3 \cdot \tfrac{8}{27}\right] = \tfrac{4\pi}{9} = 1.396$$

$4: \begin{cases} 1: \text{limits} \\ 2: \text{integrand} \\ 1: \text{answer} \end{cases}$

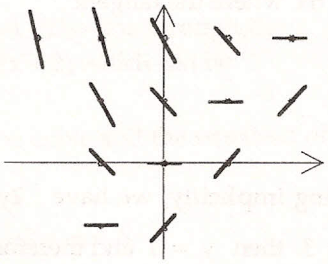
Exam I
Section II
Part B — No Calculators

4. p. 22

(a) We calculate slopes at each of the fourteen points.

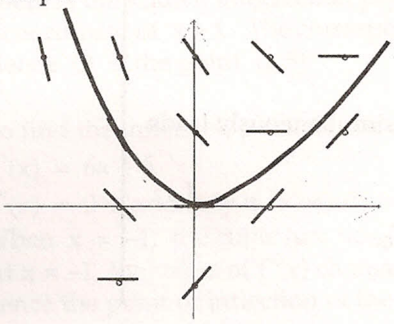
At (–2,2), m = –4. At (–1,2), m = –3.
At (0,2), m = –2. At (1,2), m = –1.
At (2,2), m = 0. At (–1,1), m = –2.
At (0,1), m = –1. At (1,1), m = 0.
At (2,1), m = 1. At (0,–1), m = 1.
At (0,0), m = 0. At (0,1), m = –1.
At (–1,–1), m = 0. At (0,–1), m = 1.

Then draw short line segments through each of the points with the appropriate slope.



$2:\begin{cases} 1:\text{zero slopes} \\ 1:\text{nonzero slopes} \end{cases}$

(b)



Solution curve must

$2:\begin{cases} 1:\text{go through } (-1,1); \\ 1:\text{follow the given slope lines and extend to the boundary of the slope field.} \end{cases}$

(c) At the point (1,0), $\frac{dy}{dx} = 1 - 0 = 1$. Hence the slope of the straight line solution must be $m = 1$. The line through the point (1,0) with slope $m = 1$ is $y - 0 = 1(x - 1)$.
Hence the solution is $y = x - 1$.

$2:\begin{cases} 1:\text{slope} \\ 1:\text{tangent equation} \end{cases}$

(d) Given the function $y = x - 1 + Ce^{-x}$, we have $\frac{dy}{dx} = 1 - Ce^{-x}$.
We can also write the expression $x - y$ in terms of x:
$x - y = x - \left(x - 1 + Ce^{-x}\right)$.
This simplifies to $x - y = 1 - Ce^{-x}$.
Thus, if $y = x - 1 + Ce^{-x}$, then $\frac{dy}{dx} = x - y$.

$3:\begin{cases} 1:\frac{dy}{dx} = 1 - Ce^{-x} \\ 1:\text{substitution} \\ 1:\text{conclusion} \end{cases}$

5. p. 23

(a) $f'(3) = 2$. Hence the slope of the tangent line at the point $(3,1)$ is $m = 2$. Then an equation of the tangent line (in point-slope form) is:

$$y - 1 = 2(x - 3).$$

$2: \begin{cases} 1: \text{slope} \\ 1: \text{tangent equation} \end{cases}$

(b) f has critical values at the points where $x = 1$ and $x = -3$, because $f'(x) = 0$.
To the immediate left of $x = 1$, $f'(x) < 0$, implying f is decreasing there.
To the immediate right of $x = 1$, $f'(x) > 0$, implying f is increasing there.
Since f is decreasing to the left of $x = 1$ and increasing to the right of $x = 1$, there is a local minimum there. Both to the left and right of $x = -3$, $f'(x) < 0$, so there is no relative max/min there.

$2: \begin{cases} 1: \text{answer} \\ 1: \text{justification} \end{cases}$

(c) $f''(2)$ is the slope of the graph of $f'(x)$ at $x = 2$. Draw an estimate for the tangent line to $f'(x)$ at $x = 2$. Pick two points, such as $(1.2, 1)$ and $(3, 2.5)$, and the slope is $\frac{2.5 - 1}{3 - 1.2} = \frac{1.5}{1.8} = \frac{5}{6}$. (Any answer between 0.5 and 1.25 would be satisfactory.)

1: answer

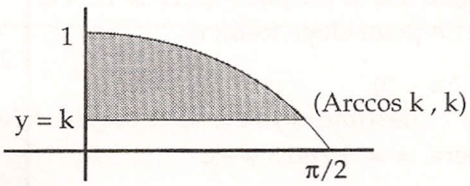
(d) f has an inflection point wherever f' has a relative extreme point. This occurs at $x = -3, -1, 3$.

$2: \begin{cases} 1: \text{answer} \\ 1: \text{justification} \end{cases}$

(e) The only candidates for maximum value are the endpoints $x = 0$ and $x = 4$, and the critical number at $x = 1$. In part (b) it was established that f has a local minimum at the $x = 1$. So the maximum value occurs at an endpoint. At $x = 0$, $f(0) = \int_0^0 f'(x)\,dx = 0$. Since the area of the region below the x-axis is smaller the the area of the region above the x-axis, $f(4) = \int_0^4 f'(x)\,dx > 0$. Hence f has its maximum value for that interval at the right-hand endpoint, $x = 4$.

2: answer

6. p. 24



(a)

$$\text{area} = \int_0^{\text{Arccos } k} (\cos x - k)\,dx = \left[\sin x - kx\right]_0^{\text{Arccos } k}$$

$$= \sin(\text{Arccos } k) - k \,\text{Arccos } k \quad \left\{ \begin{array}{l} \text{Note: Letting } A = \text{Arccos } k, \text{ we} \\ \text{have } \cos A = k \text{ and} \\ \sin A = \sqrt{1-\cos^2 x} = \sqrt{1-k^2}. \end{array} \right.$$

$$= \sqrt{1-k^2} - k\,\text{Arccos } k$$

3: { 1: limits, 1: integrand, 1: answer }

(b) $k = \frac{1}{2} \quad \Rightarrow \quad A = \frac{\sqrt{3}}{2} - \frac{1}{2}\,\text{Arccos}\frac{1}{2} = \frac{\sqrt{3}}{2} - \frac{\pi}{6} \approx 0.342$

2: answer

(c) In general, $A = \sqrt{1-k^2} - k\,\text{Arccos } k.$

$$\text{Then } \frac{dA}{dt} = \frac{-k\frac{dk}{dt}}{\sqrt{1-k^2}} - (\text{Arccos } k)\frac{dk}{dt} - k \cdot \frac{-1}{\sqrt{1-k^2}}\frac{dk}{dt}$$

$$= \frac{dk}{dt}\left[\frac{-k}{\sqrt{1-k^2}} - \text{Arccos } k + \frac{k}{\sqrt{1-k^2}}\right]$$

$$= (-\text{Arccos } k)\frac{dk}{dt}.$$

With $k = \frac{1}{2}$ and $\frac{dk}{dt} = \frac{1}{\pi}$, we obtain $\frac{dA}{dt} = -\frac{\pi}{3} \cdot \frac{1}{\pi} = -\frac{1}{3}$.

4: { 2: $\frac{dA}{dt}$, 1: substitution, 1: answer }